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Toward the rectilinear crossing number of K_n: new drawings, upper bounds, and asymptotics

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Abstract

Scheinerman and Wilf (Amer. Math. Monthly 101 (1994) 939) assert that "an important open problem in the study of graph embeddings is to determine the rectilinear crossing number of the complete graph K_n". A *rectilinear drawing of* K_n is an arrangement of n vertices in the plane, every pair of which is connected by an edge that is a line segment. We assume that no three vertices are collinear, and that no three edges intersect in a point unless that point is an endpoint of all three. The *rectilinear crossing number of* K_n is the fewest number of edge crossings attainable over all rectilinear drawings of K_n. For each n we construct a rectilinear drawing of K_n that has the fewest number of edge crossings and the best asymptotics known to date. Moreover, we give some alternative infinite families of drawings of K_n with good asymptotics. Finally, we mention some old and new open problems.
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Keywords: Crossing number; Rectilinear; Complete graph

1. Introduction and history

Given an arbitrary graph G, determining a drawing of G in the plane that produces the fewest number of edge crossings is NP-Complete [9]. The complexity is not known for an arbitrary graph when the edges are assumed to be line segments [2]. Recent

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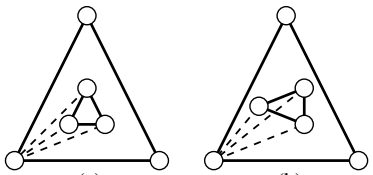


Fig. 1. Concentric versus non-concentric triangles.

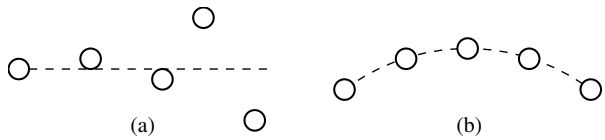


Fig. 2. Positioning vertices using Jensen's [15] and Hayward's [14] constructions.

exciting work on the general crossing number problem (where edges are simply homeomorphs of the unit interval $[0,1]$ rather than line segments) has been accomplished by Pach et al. [18], who give a tight lower bound for the crossing number of families of graphs with certain forbidden subgraphs. We study the specific instance of determining the *rectilinear crossing number of* K_n, denoted $\overline{\mathrm{cr}}(K_n)$, and we offer drawings with "few" edge crossings. The difficulty of determining the exact value of $\overline{\mathrm{cr}}(K_n)$, even for small values of n, manifests itself in the sparsity of literature [5,7,12,21]. Other contributions are given as general constructions [14,15] that yield upper bounds and asymptotics, none of which lead to exact values of $\overline{\mathrm{cr}}(K_n)$ for all n. Finally, there is an elegant and surprising connection between the asymptotics of the rectilinear crossing number of K_n and Sylvester's four-point problem of geometric probability [20,24].

Much of the information regarding progress of any kind has been disseminated by personal communication, and now in this era of "the information highway", some revealing sources of the unfolding story can be found on the web [1].

In this paper we offer new constructions, upper bounds, and asymptotics, which we motivate and explain by the interesting and non-deterministic historical progress of the problem and its elusive solution.

2. Definitions

2.1. Concentric triangles

Upon examining different configurations of vertices in the plane, one quickly realizes that drawings that minimize crossings tend to have vertices aligned along three axes, forming a triangular structure of nested concentric triangles; such configurations are "opposite" in flavour to placing vertices on a convex hull. Two nested triangles t_1 and t_2 are *concentric* if and only if any edge with endpoints in t_1 and t_2 does not intersect any edge of t_1 or t_2 (see Fig. 1). In K_4 through K_{10}, for which optimal drawings are known [3,12,23], the tripartite pattern is evident. The same pattern exists in generalized constructions presented by Jensen [15] and Hayward [14] for any K_n.

Various schemes are possible for positioning vertices within each of the three parts. In Jensen's construction, vertices along an axis are positioned by alternating above and below the axis (see Fig. 2a). In Hayward's construction, vertices along an axis are positioned on a concave curve (see Fig. 2b). Alternatively, the collection of vertices

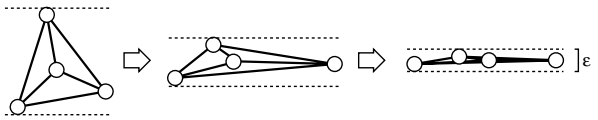


Fig. 3. Flattening a clustervertex.

along each axis could be arranged to minimize crossings within the collection, while maintaining concentricity of the triangles. We examine a construction and variations, originally suggested by Singer [21], that positions vertices along each axis by recursive definition of similarly constructed smaller graphs.

2.2. Clustervertices and clusteredges

We identify specific sets of edges, sets of vertices, and subgraphs, within the larger construction of K_n. Those components of the graph that are recursively defined form *clustervertices*. Each clustervertex is itself a complete graph K_a, where $a<n$; a clustervertex with a vertices is said to have *order* a. If both endpoints of an edge uw are contained within clustervertex c, then uw is *internal* to c. Similarly, a vertex w contained within a clustervertex c is internal to c. Given two clustervertices c_1 and c_2, the set of all edges that have one endpoint in each of c_1 and c_2 form a *clusteredge*. Clustervertex or clusteredge a *intersects* clustervertex or clusteredge b if there exist edges $e_1 \in a$ and $e_2 \in b$ such that e_1 and e_2 cross. Finally, if q clusteredges meet at clustervertex c, then c has *clusterdegree* q.

2.3. Flattening a clustervertex

Recursively constructed clustervertices are *flattened* by an affine transformation [16, Chapter 15]; the number of edge crossings remains unchanged under any affine transformation. Vertices appear as a sequence of nearly collinear vertices. Of course, no three vertices in the graph can be collinear, thus the flattened clustervertex has some height $\varepsilon>0$ (see Fig. 3) and its edge crossings are unaltered by the transformation.

Formally, we flatten clustervertex c as follows. Choose any line l_0 that is neither parallel nor perpendicular to any edge of c. Clustervertex c is flattened by a scaling along $l_0^{\perp}$. When l_0 coincides with the x-axis, this scaling can be defined by $(x, y) \to (x, \gamma y)$ for a constant $0<\gamma<1$. Each edge e_i of c determines a line, l_i, and each line, in turn, determines two open half-planes, $l_{i,\mathrm{top}}$ and $l_{i,\mathrm{bot}}$, such that $\partial l_{i,\mathrm{top}} = \partial l_{i,\mathrm{bot}} = l_i$ (where ∂X denotes the boundary of region X). The labelling of each half-plane is determined as follows. The two open half-planes of l_0 are assigned arbitrary orientations, top and bottom. Since l_0 is not parallel to any edge e_i, l_0 must intersect every line l_i; call each such point of intersection p_i.

Let $l_i^{\perp}$ be the line perpendicular to l_i that passes through p_i. Let q_i and r_i be the two unique points that lie on $l_i^{\perp}$ and are unit distance away from p_i. One of these points must lie in $l_{0,\mathrm{top}}$ and the other in $l_{0,\mathrm{bot}}$ (see Fig. 4). The side of l_i on which

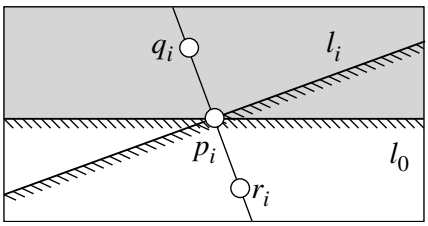


Fig. 4. $q_i \in e_{i,\text{top}}$.

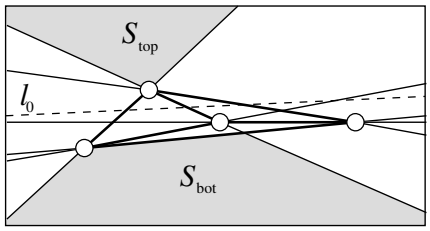


Fig. 5. Sides of a clustervertex.

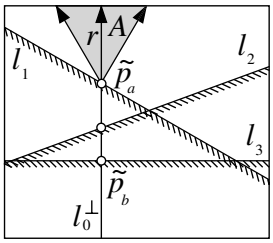


Fig. 6. Lemma 1.

each lies defines $l_{i,\text{top}}$ and $l_{i,\text{bot}}$. The flattened clustervertex c has two *sides*, S_{top} and S_{bot} (see Fig. 5) that are defined by

$$S_{\text{top}} = \bigcap_i l_{i,\text{top}} \quad \text{and} \quad S_{\text{bot}} = \bigcap_i l_{i,\text{bot}}. \tag{1}$$

Lemma 1. *A side S of a flattened clustervertex c contains a non-empty and unbounded region.*

Proof. Let k be the number of edges in c. Take any line $l_0^{\perp}$ perpendicular to l_0 such that $l_0^{\perp}$ intersects the lines $l_1,\ldots,l_k$ at k distinct points; name these points $\tilde{p}_1,\ldots,\tilde{p}_k$, respectively (see Fig. 6). These points can be ordered along the line $l_0^{\perp}$. Let $a,b \in \{1,\ldots,k\}$ be such that $\tilde{p}_a$ and $\tilde{p}_b$ are the endpoints in this ordering. Consider the ray r_i defined by $l_0^{\perp} \cap l_{i,\text{top}}$. Clearly, $\bigcap_i r_i \subseteq S_{\text{top}}$. Thus, either $\bigcap_i r_i = r_a$ or $\bigcap_i r_i = r_b$. Therefore, either $r_a \subseteq S_{\text{top}}$ or $r_b \subseteq S_{\text{top}}$. Whichever lies in S_{top}, call this ray r.

Let $\sigma(l_i)$ be the slope of line l_i with respect to l_0 and let $s = \max_i(|\sigma(l_i)|)$. Let A be the region defined by a sector bisected by r whose edges have slopes s and $-s$. Since r is the last point of intersection of any l_i along $l_0^{\perp}$ and since $|s| \geqslant |\sigma(l_i)|$ for all i, no line l_i intersects the region A. Since $r \subseteq S_{\text{top}}$ and A does not intersect any of the half-plane boundaries l_i, $A \subseteq S_{\text{top}}$. Since $s \neq -s$, A is a non-empty region. Furthermore, since ray r is contained in A, the region A is unbounded. An identical argument holds for S_{bot}. □

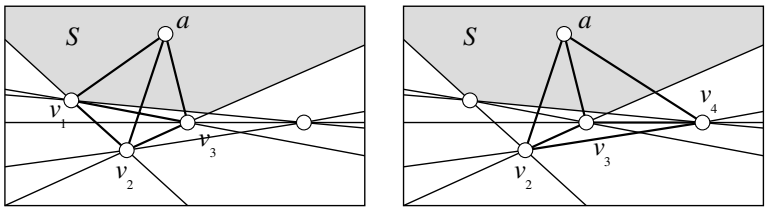


Fig. 7. No matter where a is located in S, the crossings in the K_4 formed by a and any three vertices from c remain constant. (a) The K_4 induced by $\{a, v_1, v_2, v_3\}$ contains one crossing, and (b) the K_4 induced by $\{a, v_2, v_3, v_4\}$ does not contain any crossings.

Lemma 2. *Let c be a flattened clustervertex and let S be one of the sides of c. Given any two points $a, a' \in S$, the number of crossings created by the edges from a to the vertices of c and the edges from a' to the vertices of c are identical.*

Proof. Choose any three vertices v_1, v_2, and v_3 from c. Choose any a and $a' \in S$. Let K and K' denote the K_4 subgraphs induced by $\{a, v_1, v_2, v_3\}$ and $\{a', v_1, v_2, v_3\}$, respectively. Since region S is defined by extensions of the edges of c, points a and a' lie on the same side of any edge $e_i \in c$. Specifically, a and a' reside on the same sides of the lines associated with the edges v_1v_2, v_2v_3, and v_1v_3. However, for the crossing count to be different in K and K', points a and a' must reside on opposite sides of at least one of these three lines. Thus, K contains a crossing if and only if K' contains a crossing (see Fig. 7). Every crossing for which a and a' are responsible can be described by such a K_4. Therefore, for any a and a' in S, the number of crossings remains constant. □

The *height* of clustervertex c with respect to a line l_0 is the magnitude of the smallest interval that contains all vertices of c along the projection of c onto $l_0^{\perp}$.

Lemma 3. *Let c be a clustervertex flattened by scaling along the axis $l_0^{\perp}$ and suppose S is a side of c. Let θ be the maximum acute angle between l_0 and any edge along the boundary of S. For every angle $0 < \alpha < \pi/2$, there exists an $\varepsilon > 0$ such that when clustervertex c is flattened to height ε, $\theta \leqslant \alpha$.*

Proof. When the c is scaled, its height is reduced from ε to ε'. Let $\gamma = \varepsilon'/\varepsilon$. The boundary of S is determined by a collection of rays and line segments contained in $\bigcup_i l_i$. Every e_i can be defined in terms of a horizontal component, x_i, and a vertical component, y_i, with respect to coordinate axes l_0 and $l_0^{\perp}$ (see Fig. 8). When c is scaled, the vertical component is reduced to $y_i' = \gamma y_i$. The angle ϕ_i between l_i and l_0 is reduced to $\phi_i' = \mathrm{Arctan}(y_i'/x_i)$. Since $\mathrm{Arctan}(x)$ is a continuous function and $\mathrm{Arctan}(0) = 0$, therefore, $\forall z_i = y_i/x_i$, $\forall \phi_i' > 0, \exists \gamma_i > 0$ such that $\mathrm{Arctan}(\gamma_i z_i) < \phi_i'$. This must be true for $\phi_i' = \alpha$. Let $\gamma = \min_i \gamma_i$. Thus, when c is flattened to height $\varepsilon' = \gamma\varepsilon$, the angle ϕ_i' of every edge e_i is guaranteed to be less than or equal to α. □

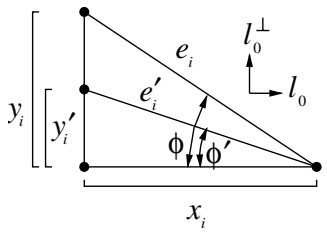


Fig. 8. Lemma 3.

Lemma 3 implies that any clustervertex c can be flattened to an arbitrarily small height such that its sides S_{top} and S_{bot} each contain a sector whose angle is arbitrarily close to π.

A clusteredge incident on clustervertices c_1 and c_2 is said to *dock* on side S of c_1 whenever all vertices of c_2 lie within S. Given a flattened clustervertex c and two clusteredges h_1 and h_2 such that h_1 docks on side S_{top} of c and h_2 docks on side S_{bot} of c, no edge crossings exist between h_1 and h_2. When two clusteredges h_1 and h_2 dock on the same side of a clustervertex c, we say h_1 and h_2 *merge* at c (see Fig. 12).

3. Counting toolbox

Given a generalized definition for graph construction involving clustervertex interconnection, the following functions count edge crossings for the various types of edge intersections.

3.1. $f(k)$: Clusteredge from single vertex to clustervertex

When a new vertex u is created, new edges are added from u to all other existing vertices. Specifically, given a clustervertex c of order k such that u lies within side S of c, an edge must be added from u to every vertex in c. An edge from u to a vertex w in c may cross some internal edges of c. The projection of the vertices of c onto l_0 gives an ordering of the vertices. If w is the ith vertex in the sequence of vertices of c, $i-1$ vertices lie on one side of w in c and $k-i$ vertices lie on the other side (see Fig. 9a). Thus, edge uw must cross at most $(i-1)(k-i)$ edges of c. If we add edges from u to every vertex in c, the number of new edge crossings within c will be at most

$$f(k)=\sum_{i=1}^{k}(i-1)(k-i)=\frac{k^3}{6}-\frac{k^2}{2}+\frac{k}{3}. \tag{2}$$

If we add two vertices v_1 and v_2 on opposite sides of a clustervertex c, then for every internal vertex w of c, the internal edges that span w will be crossed exactly once, either by edge v_1w or by edge v_2w but not both (see Fig. 9b). The number of new edge crossings among vertices of c and v_1 and v_2 will be exactly $f(k)$.

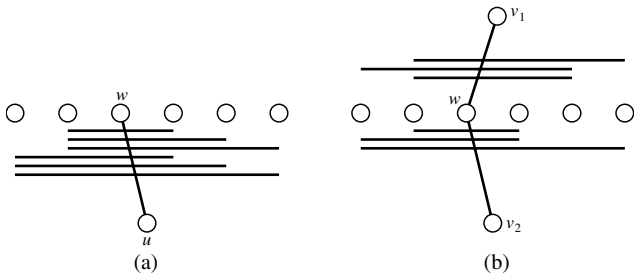


Fig. 9. Edge uw crosses at most six internal edges.

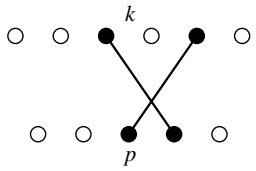


Fig. 10. $\binom{p}{2}\binom{k}{2}$ crossings.

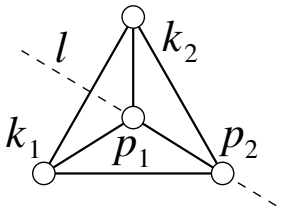


Fig. 11. Convex quadrilateral.

3.2. $i(p,k)$: Internal clusteredge intersections

Given two clustervertices c_k and c_p of orders k and p, and a clusteredge e between them that docks completely on one side of each clustervertex, selecting two vertices from each clustervertex forms a convex quadrilateral that contributes one edge crossing (see Fig. 10). The number of edge crossings within e is given by

$$i(p,k) = \binom{p}{2}\binom{k}{2} = \frac{p(p-1)k(k-1)}{4}. \tag{3}$$

The function $i(p,k)$ gives an exact count of the number of edge crossings because any quadrilateral constructed from two vertices from c_p and two vertices from c_k must be convex. To see why, assume by way of contradiction that $\{k_1, k_2, p_1, p_2\}$ forms a non-convex quadrilateral Q. Three points must lie on the convex hull of Q and the fourth must lie in the interior of Q. Without loss of generality, assume p_1 lies in the interior of Q (see Fig. 11). Points k_1 and k_2 must lie on opposite sides of the line l determined by edge p_1p_2. By the definition of side, line l cannot intersect a side of c_p. Therefore, k_1 and k_2 cannot reside within the same side of c_k.

3.3. $e(k,p,j)$: Two clusteredges merge at a clustervertex

Let c_j, c_k, and c_p be clustervertices of orders j, k, and p, respectively, such that c_p and c_j reside within a side S of c_k. Let e_p be a clusteredge between c_p and c_k and let e_j be a clusteredge between c_j and c_k such that e_p and e_j merge at c_k (see Fig. 12).

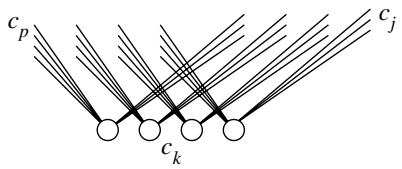


Fig. 12. Two clusteredges merge at a clustervertex.

Assume e_p does not intersect c_j and e_j does not intersect c_p. The number of crossings between edges of e_p and e_j (ignoring crossings with edges internal to clustervertex c_k) is given by

$$e(k,p,j)=\sum_{i=0}^{k-1} ipj=\frac{pjk(k-1)}{2}. \tag{4}$$

If the two clusteredges intersect away from a clustervertex, then the number of crossings is simply $p\cdot j\cdot k\cdot l$, where the clusteredge crossing is between four clustervertices of orders p, j, k, and l.

4. Recursive definitions of K_n

The following constructions of K_n involve recursive definition by connecting q clustervertices K_k of order k, where $n=q\cdot k$. Scheinerman and Wilf show that $\overline{\mathrm{cr}}(K_n)=\Theta(n^4)$ and that $\lim_{n\to\infty}\overline{\mathrm{cr}}(K_n)/n^4$ exists [20]. In a worst case drawing, where edge crossings are maximized, every subset of four vertices contributes one edge crossing. This occurs when all vertices lie on a convex hull, creating $\binom{n}{4}$ crossings. Thus, when a better drawing is found, we examine what fraction of the crossings remain by taking the limit of $g(n)/\binom{n}{4}$ as $n\to\infty$, where $g(n)$ is a count of the crossings in the new drawing.

4.1. Triangular definition

Singer suggests a recursive construction [21,24] where, given $n=3^j$, we draw K_n by taking three flattened instances of $K_{n/3}$, denoted by a, b, and c, and adding new clusteredges (see Fig. 13). Each instance of $K_{n/3}$ is drawn recursively. K_3 gives a base case. By Lemma 3, sufficiently flattening each clustervertex $x\in\{a,b,c\}$ ensures that the remaining two clustervertices lie within opposite sides of x.

Let $k=n/3$ and let $C_3(n)$ represent the total number of crossings in K_n under the drawing defined by this recursive construction. There are $C_3(k)$ crossings internal to each of the clustervertices, $k\cdot f(k)$ crossings for each clustervertex corresponding to clusteredge to clustervertex dockings, and $i(k,k)$ crossings internal to each clusteredge.

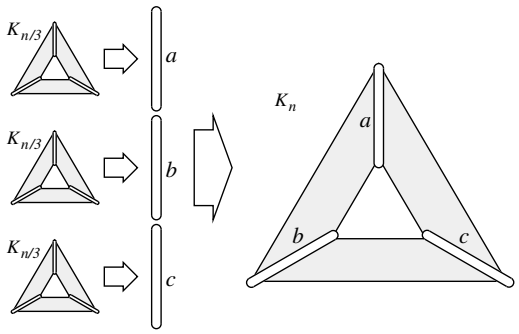


Fig. 13. K_n defined by three $K_{n/3}$.

Given that $C_3(3)=0$, the total number of crossings is given by

$$C_3(n)=3C_3(k)+3k\cdot f(k)+3i(k,k)$$

$$=\frac{5}{312}n^4-\frac{1}{8}n^3+\frac{7}{24}n^2-\frac{19}{104}n \tag{5}$$

$$\Rightarrow \lim_{n\to\infty}\frac{C_3(n)}{\binom{n}{4}}=\frac{5}{13}\approx 0.3846. \tag{6}$$

4.2. Recursive definitions using a larger K_a

Just as we do for K_3, we may use any optimal drawing of K_a as a recursive template. Given $n=a^j$ and $k=n/a$, we apply an analogous procedure where clustervertices are defined recursively. In addition to counting recursive terms, $C_a(k)$, internal clusteredge crossings, $i(k,k)$, and clusteredge–clustervertex crossings, $k\cdot f(k)$, we must also count pairs of clusteredges that merge, $e(k,k,k)$, and clusteredge crossings away from a clustervertex, k^4. Using K_4 as a basis[2] and $C_4(4)=0$, we derive

$$C_4(n)=4C_4(k)+6i(k,k)+6k\cdot f(k)+4e(k,k,k)$$

$$=\frac{1}{56}n^4-\frac{2}{15}n^3+\frac{7}{24}n^2-\frac{37}{210}n \tag{7}$$

$$\Rightarrow \lim_{n\to\infty}\frac{C_4(n)}{\binom{n}{4}}=\frac{3}{7}\approx 0.4286. \tag{8}$$

[2] Since K_4 has an even vertex count, counting crossings using $f(k)$ to count crossings at dockings requires explicit pairing of clusteredges with the top and bottom of clustervertices. In doing so, exactly $k\cdot f(k)+\mathrm{i}(k,k)$ crossings are associated with every clusteredge.

Table 1
Asymptotic counts for $C_a(n)$ compared with known bounds

	a	$\lim_{n\to\infty} g(n)/\binom{n}{4}$	Comment
Singer [21]	3	0.3846	$n=3^j$, $C_3(3)=0$
Brodsky et al.	4	0.4286	$n=4^j$, $C_4(4)=0$
Brodsky et al.	5	0.3935	$n=5^j$, $C_5(5)=1$
Brodsky et al.	7	0.3885	$n=7^j$, $C_7(7)=9$
Brodsky et al.	9	0.3846	$n=9^j$, $C_9(9)=36$
Jensen [15]	—	0.3888	Any n
Hayward [14]	—	0.4074	Any n
Scheinerman and Wilf [20]	—	0.2905	Lower bound
Guy [11]	—	0.3750	Conjectured $\overline{\mathrm{cr}}(K_n)$ (non-rectilinear)

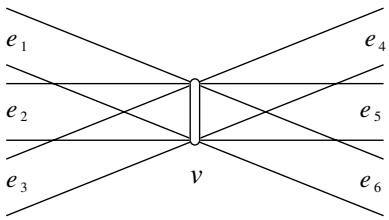


Fig. 14. Balanced clusteredge dockings.

Using K_5 as a basis and $C_5(5)=1$, we derive

$$C_5(n) = 5C_5(k) + 10i(k,k) + 10k\cdot f(k) + 10e(k,k,k) + k^4$$

$$= \frac{61}{3720}n^4 - \frac{1}{8}n^3 + \frac{7}{24}n^2 - \frac{227}{1240}n \tag{9}$$

$$\Rightarrow \lim_{n\to\infty}\frac{C_5(n)}{\binom{n}{4}} = \frac{227}{155} \approx 0.3935. \tag{10}$$

Similarly, we derive limits using K_7 and K_9 as templates (see Table 1).

As one would expect, the limit for K_9 is equal to that for K_3, since both are powers of three. For any odd a, we derive a generalized exact count using a recursive K_a construction. We require a count for the number of crossings in K_a, both for our base case, $C_a(a)=\overline{\mathrm{cr}}(K_a)$, and for recursively-defined clusteredge to clusteredge crossings.

The count is calculated as follows. Let $k=n/a$. We take a recursive instances of K_k which contribute $a\cdot C_a(k)$ crossings. We add crossings for every pair of clusteredges that merge at a clustervertex. Each clustervertex has clusterdegree $a-1$. To minimize crossings, clusteredges must be split evenly on either side of a clustervertex v (see Fig. 14). No matter how large a is, v can be flattened such that exactly $(a-1)/2$ clusteredges dock on each side. Thus, clusteredge dockings contribute $2a\binom{(a-1)/2}{2}e(k,k,k)$

crossings. Pairs of dockings on opposite sides of a clustervertex contribute exactly $\binom{a}{2}k \cdot f(k)$ crossings.

Clusteredges have internal crossings that add another $\binom{a}{2}i(k,k)$. Finally, we must account for clusteredge to clusteredge crossings that occur in K_a itself; thus we add $\overline{\mathrm{cr}}(K_a) \cdot k^4$. This gives

$$C_a(n) = a \cdot C_a(k) + \binom{a}{2} k \cdot f(k) + 2a \binom{\frac{a-1}{2}}{2} e(k,k,k) + \binom{a}{2} i(k,k) + \overline{\mathrm{cr}}(K_a) \cdot k^4. \tag{11}$$

We can solve for a non-recursive closed form of $C_a(n)$ by simplifying

$$C_a(n) = \frac{n}{a}\overline{\mathrm{cr}}(K_a) + \sum_{j=1}^{\log_a n - 1} a^{j-1} \left[\binom{a}{2} k \cdot f(k) + \binom{a}{2} i(k,k) + 2a \binom{\frac{a-1}{2}}{2} e(k,k,k) + \overline{\mathrm{cr}}(K_a) \cdot k^4 \right], \tag{12}$$

where $k = n/a^j$.

Of all recursive constructions for which $\overline{\mathrm{cr}}(K_a)$ is known, the best results are achieved by $C_3(n)$ and $C_9(n)$ (see Table 1). The construction can easily be generalized (for n not necessarily divisible by 3) by partitioning n into three parts of sizes $\lfloor n/3 \rfloor$, $\lceil n/3 \rceil$, and $n - \lfloor n/3 \rfloor - \lceil n/3 \rceil$. Since two of the three parts will always have the same size, $f(k)$ always gives an exact count. Let $C_g(n)$ denote a count of the crossings in the generalized construction of $C_3(n)$ given by

$$C_g(n) = \sum_{x \in \{1,2,3\}} [C_g(k_x) + f(k_x)] + i(k_1,k_2) + i(k_2,k_3) + i(k_1,k_3), \tag{13}$$

where $k_1 = \lfloor n/3 \rfloor$, $k_2 = \lceil n/3 \rceil$, and $k_3 = n - k_1 - k_2$. By induction, one can show that $C_g(n) < \mathrm{jen}(n)$ for $n \geqslant 24$, where $\mathrm{jen}(n)$ is the number of crossings in K_n using Jensen's construction [15]. [3] Thus, asymptotically,

$$C_g(n) < 3 \cdot [\mathrm{jen}(k) + k \cdot f(k) + i(k,k)], \quad \text{with } k = n/3 \tag{14}$$

and we get an upper bound of 0.3848 for a general n. In the next section we offer some improvements.

5. Asymptotic improvements

Within the recursive constructions presented thus far, edges arriving at a flattened clustervertex are balanced; if q edges arrive at clustervertex c of degree p, then exactly

[3] $\mathrm{jen}(n) = \lfloor \frac{7n^4 - 56n^3 + 128n^2 + 48n\lfloor (n-7)/3 \rfloor + 108}{432} \rfloor$.

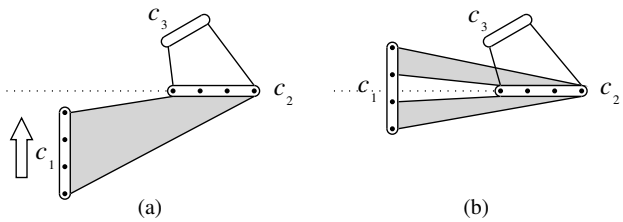


Fig. 15. Sliding a clustervertex.

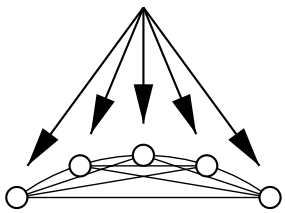


Fig. 16. Minimizing crossings from above.

$q/2$ edges arrive at c from each side and $(q/2)f(p)$ crossings are added. However, depending on the side of entry, the number of edges crossed when entering a clustervertex differs. Thus, it may be advantageous to have an imbalance in the number of edges docking on each side of a clustervertex.

Most of the crossings in $C_3(n)$ occur at the top level of the recurrence, as is shown by

$$\lim_{n\to\infty} \frac{C_3(n) - 3C_3(n/3)}{C_3(n)} = \frac{26}{27}. \tag{15}$$

Improving the top level of the construction while slightly compromising on recursive constructions could reduce the total crossings. Improvements at the top level can be achieved by moving a clustervertex c_1 to alter the number of edges that reach a neighbouring clustervertex c_2 from above and below (see Fig. 15). Lemma 3 allows us to flatten c_2 such that, upon sliding c_1, any two vertices of c_1 can be made to lie on opposite sides of c_2. In doing so, however, new crossings are created at the merging of clusteredges from c_1 and c_3. Thus, there exists a point of balance that minimizes total crossings lost and gained by the translation.

5.1. Maximally asymmetric internal clustervertices

In the extreme case, we construct each of the three partitions by taking a convex K_k (see Figs. 16 and 18). Crossings from above are minimized and crossings from below are maximized to form a maximally asymmetric drawing.

Let $k = n/3$ and let $a + b = k$ determine how much to slide the clustervertex, where b is a measure of how many vertices in one clustervertex change position relative to the other two. Assuming each clustervertex is moved by the same amount, the top-level

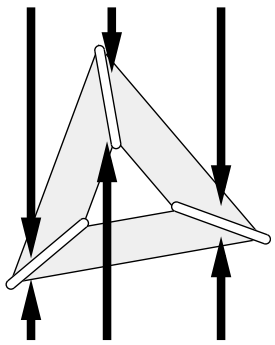


Fig. 17. Docking above versus below.

graph will appear as in Fig. 18. Accounting using the usual tools gives the following count of crossings

$$C_{\mathrm{m}}(n,a) = 3\left[\binom{k}{4} + a\cdot f(k) + \mathrm{i}(a,a) + i(b,b) + 2i(a,b) + e(a,b,b)\right.$$

$$\left. + 2e(a,a,b) + e(b,b,b) + 2e(b,a,b) + ab^3 + a^2b^2\right] \tag{16}$$

$$= \frac{19}{648}n^4 - \frac{5}{54}n^3a + \frac{1}{6}n^2a^2 - \frac{5}{36}n^3$$

$$+\frac{1}{6}n^2a - \frac{1}{2}na^2 + \frac{17}{72}n^2 + \frac{1}{3}na - \frac{1}{4}n. \tag{17}$$

$C_{\mathrm{m}}(n,a)$ is a quadratic polynomial in a and is minimized when $a_0 = 5n/18 + \frac{1}{3}$. This gives

$$C_{\mathrm{m}}(n,a_0) = \frac{4}{243}n^4 - \frac{85}{648}n^3 + \frac{67}{216}n^2 - \frac{7}{36}n \tag{18}$$

$$\Rightarrow \lim_{n\to\infty}\frac{C_{\mathrm{m}}(n,a_0)}{\binom{n}{4}} = \frac{32}{81} \approx 0.3951. \tag{19}$$

Unfortunately, $C_3(n)$ still performs better than $C_{\mathrm{m}}(n,a)$ for any a. Thus, using convex K_k as first-level clustervertices overcompensates the savings of the recursive structure in $C_3(n)$. Therefore, we define a new construction that maintains the recursive structure of $C_3(n)$ for clustervertices.

5.2. Retaining $C_3(n)$ as internal clustervertices

Previously, $f(k)$ counted access into an internal clustervertex c of order k, where dockings were balanced on both sides of c. For imbalanced access, we derive a separate

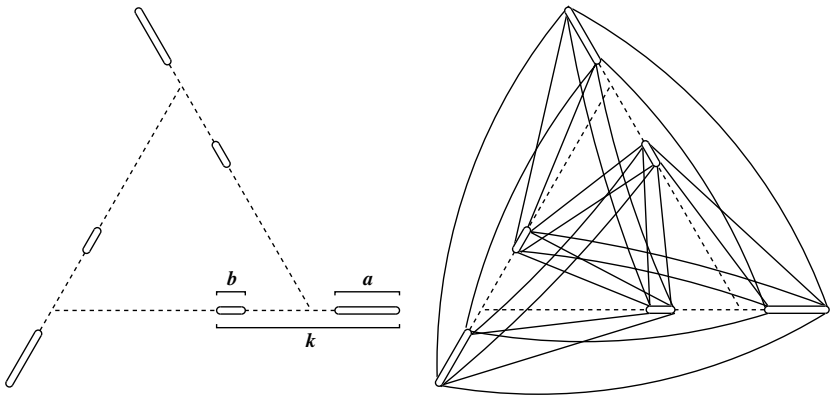


Fig. 18. Clustervertices are not actually broken, only translated; they are drawn as two parts for counting. Clusteredges are drawn as arcs to reduce clutter.

count of edge crossings entering c from above and from below where c is recursively defined by $C_3(k)$ and $k = 3^j$. In the base cases, $n = 3$, no crossings occur above and a single crossing occurs below. Thus, we define $f_{\text{top}}(3) = 0$ and $f_{\text{bot}}(3) = 1$. Assume the triangles are arranged recursively to point upwards. We count crossings as follows. Assume $k = n/3$. If the new point is positioned above the clustervertex, $3f_{\text{top}}(k)$ edges are crossed recursively and $3e(k,k,1)$ are crossed at the top level. If the new point is positioned below the clustervertex, then k^3 additional crossings occur (see Fig. 17). Thus, we derive the following recurrences:

$$f_{\text{top}}(n) = 3[f_{\text{top}}(k) + e(k,k,1)] = \frac{n^3}{16} - \frac{n^2}{4} + \frac{3n}{16}, \tag{20}$$

$$f_{\text{bot}}(n) = 3[f_{\text{bot}}(k) + e(k,k,1)] + k^3 = \frac{5n^3}{48} - \frac{n^2}{4} + \frac{7n}{48}. \tag{21}$$

As expected, $f(n) = f_{\text{top}}(n) + f_{\text{bot}}(n)$. The difference between $f_{\text{top}}(n)$ and $f_{\text{bot}}(n)$ is significant as is shown by

$$\lim_{n\to\infty} \frac{f_{\text{top}}(n)}{f_{\text{bot}}(n)} = \frac{3}{5}. \tag{22}$$

Sliding a clustervertex creates new crossings at the merging of two clusteredges and at the crossing of new clusteredges (see Fig. 18b). We count the cost of sliding one, two, or three clustervertices. These counts are given by $C_{s1}(n,a)$, $C_{s2}(n,a)$, and $C_{s3}(n,a)$, respectively. For each, a represents the portion of the affected clustervertex that still docks on the same side of incident clustervertices. The value a is defined in terms of n. When more than one clustervertex is moved, both or all three being moved are moved by the same amount (Table 2).

Table 2
Asymptotic improvements on $C_3(n)$

	Graph	Internal	Top level	Total	Minimizing a_0
Singer [21]	$C_3(n)$	0.0142	0.3704	0.3846	
Brodsky et al.	$C_m(n,a)$	0.0370	0.3580	0.3951	$a_0 = 5n/18 + 1/3$
Brodsky et al.	$C_{s1}(n,a)$	0.0142	0.3701	0.3843	$a_0 = 23n/72 - 1/24$
Brodsky et al.	$C_{s2}(n,a)$	0.0142	0.3699	0.3841	$a_0 = 23n/72 - 1/24$
Brodsky et al.	$C_{s3}(n,a)$	0.0142	0.3696	0.3838	$a_0 = 23n/72 - 1/24$

Using a counting argument identical to that for $C_m(n,a)$, we derive the following:

$$C_{s1}(n,a) = \frac{137}{6318}n^4 - \frac{23}{648}n^3a + \frac{1}{18}n^2a^2 - \frac{31}{216}n^3 + \frac{1}{9}n^2a - \frac{1}{6}na^2 + \frac{8}{27}n^2 - \frac{1}{72}na - \frac{19}{104}n, \tag{23}$$

$$C_{s2}(n,a) = \frac{691}{25272}n^4 - \frac{23}{324}n^3a + \frac{1}{9}n^2a^2 - \frac{35}{216}n^3 + \frac{2}{9}n^2a - \frac{1}{3}na^2 + \frac{65}{216}n^2 - \frac{1}{36}na - \frac{19}{104}n, \tag{24}$$

$$C_{s3}(n,a) = \frac{139}{4212}n^4 - \frac{23}{216}n^3a + \frac{1}{6}n^2a^2 - \frac{13}{72}n^3 + \frac{1}{3}n^2a - \frac{1}{2}na^2 + \frac{11}{36}n^2 - \frac{1}{24}na - \frac{19}{104}n. \tag{25}$$

Again, each count is quadratic with respect to a and n, and each is minimized when $a_0 = 23n/72 - 1/24$. The value a represents the number of vertices in a clustervertex that dock on the bottom of the clustervertex on its (counter clockwise) right side. Thus, we require a to be an integer. One observes, however, that $a_0 = 23n/72 - 1/24$ is never an integer for $n = 3^i$, but an induction argument shows that $\lceil 23n/72 - 1/24 \rceil$ is the integer nearest a_0. Let $a_1(j) = 3^j \cdot 23/72 - 1/24$ and let $a_2(j) = \lceil 3^j \cdot 23/72 - 1/24 \rceil$. Asymptotically, $C_{s3}(n,a)$ remains unaffected since

$$\forall \varepsilon > 0,\ \exists i \in \mathbf{Z}\ s.t.\ \forall j > i \quad \left| \frac{C_{s3}(3^j, a_1(j))}{\binom{3^j}{4}} - \frac{C_{s3}(3^j, a_2(j))}{\binom{3^j}{4}} \right| < \varepsilon. \tag{26}$$

To obtain the number of edge crossings for a given $n = 3^i$ and $a_0 = 23n/72 - 1/24$, simply evaluate $C_{s3}(n, \lceil a_0 \rceil)$. Thus

$$\overline{\mathrm{cr}}(K_n) \leqslant C_{s3}(n, \lceil 23n/72 - 1/24 \rceil). \tag{27}$$

Asymptotically, this value approaches $C_{s3}(n,a_0)$, which gives

$$C_{s3}(n,a_0)=\frac{6467}{404352}n^4-\frac{1297}{10368}n^3+\frac{1009}{3456}n^2-\frac{2723}{14976}n. \tag{28}$$

A similar argument holds for $C_{s1}(n,a)$ and $C_{s2}(n,a)$. Thus, we derive the following limits:

$$\lim_{n\to\infty}\frac{C_{s1}(n,a_0)}{\binom{n}{4}}=\frac{19427}{50544}\approx 0.3846, \tag{29}$$

$$\lim_{n\to\infty}\frac{C_{s2}(n,a_0)}{\binom{n}{4}}=\frac{9707}{25272}\approx 0.3841, \tag{30}$$

$$\lim_{n\to\infty}\frac{C_{s3}(n,a_0)}{\binom{n}{4}}=\frac{6467}{16848}\approx 0.3838. \tag{31}$$

5.3. Generalized upper bounds

Theorem 4.

$$\lim_{n\to\infty}\frac{\overline{\mathrm{cr}}(K_n)}{\binom{n}{4}}\leqslant\frac{6467}{16848}\approx 0.3838. \tag{32}$$

Proof. Scheinerman and Wilf show that $\overline{\mathrm{cr}}(K_n)/\binom{n}{4}$ is a non-decreasing function [20]. We know $\overline{\mathrm{cr}}(K_n)\leqslant C_{s3}(n,a_0)$ for all $n=3^i$. Therefore,

$$\lim_{n\to\infty}\frac{\overline{\mathrm{cr}}(K_n)}{\binom{n}{4}}\leqslant\lim_{n\to\infty}\frac{C_{s3}(n,a_0)}{\binom{n}{4}}=\frac{6467}{16848}. \qquad \square \tag{33}$$

As we did for $C_3(n)$, our construction for $C_{s3}(n,a)$ can be generalized by dividing n into three partitions of sizes p_1, p_2, and p_3 such that $\max_{i,j}|p_i-p_j|\leqslant 1$. Each partition then forms a clustervertex defined recursively by $C_g(p_i)$. Clustervertices are translated by an appropriate a_i that is the integer nearest $23p_i/72-1/24$. We conjecture that such constructions produce asymptotics close to those achieved in Theorem 4.

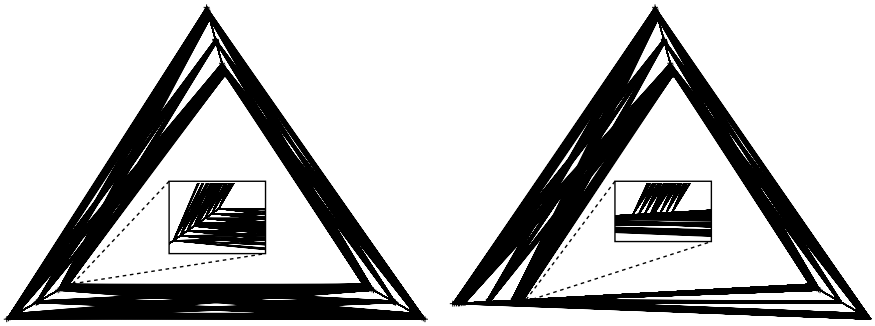
We also mention recent work on a new lower bound in Eq. (33) based on work accomplished in [5]. That is, $\overline{\mathrm{cr}}(K_{10})=62$, from which it follows that $0.3001\leqslant\frac{C_{s3}(n,a_0)}{\binom{n}{4}}$. In summary we have

$$0.3001\leqslant\frac{C_{s3}(n,a_0)}{\binom{n}{4}}\leqslant 0.3838. \tag{34}$$

5.4. Example: K_{81}

In Fig. 19, we give two rectilinear drawings of K_{81}. The first drawing is based on Singer's construction [21,24] and has 625,320 edge crossings. The second drawing[4] is based on the construction given by the strategy corresponding to $C_{s1}(81,26)=624,852$.

[4] These calculations were verified by an arbitrary precision edge crossing counter.



(a) (b)

Fig. 19. Two instances of K_{81}.

Table 3
Drawings of K_{81} that count

Strategy	$C_{s3}(81,26)$	$C_{s2}(81,26)$	$C_{s1}(81,26)$	$C_3(81)$ [21]	[15]	[14]	$\binom{81}{4}$
count	623,916	624,384	624,852	625,320	630,786	659,178	1,663,740

The largest number of edge crossings in a rectilinear drawing of K_{81} is $\binom{81}{4} =$ 1,663,740 and occurs when all 81 vertices are placed on a convex hull. The fewest number of edge crossings of K_{81} known to date is $C_{s3}(81,26) = 623,916$ (Table 3).

6. Summary and future work

In summary, most forward progress toward determining $\overline{\mathrm{cr}}(K_n)$ has been accomplished by producing a good rectilinear drawing of K_n for each n. A "good" rectilinear drawing of K_n has relatively few edge crossings and avails itself of an exact count of said crossings. Throughout the history of the problem, drawings that have produced the best asymptotic results amount to iteratively producing three clustervertices, which upon examination of the whole graph, yield a configuration of nested concentric triangles. Our best closed form and asymptotics arose from a break in tradition by yielding a graph with three clustervertices forming a set of nested triangles, but whose triangles are *not* pairwise concentric.

We offer the following open question: can one extend the technique given in Section 5 to produce a graph with more than three clustervertices that will yield better upper bounds and asymptotics for $\overline{\mathrm{cr}}(K_n)$? Singer's rectilinear drawing of K_{10} with 62 edge crossings [8,21] was the first successful recorded instance of this break with tradition. Additionally, can the technique given in Section 5 be applied successfully to other families of interesting graphs? See, for example, the work of Bienstock and Dean [3,4].

Our second open question is based on the current rapidly changing status of computing, which makes feasible the use of brute-force techniques in extracting information about small graphs. In particular, it is possible to determine the exact value of $\overline{\mathrm{cr}}(K_n)$ for small values of n beyond what is presently known [5,12,23]. For example, a complete catalogue of non-equivalent drawings is available through $n=6$ for both rectilinear and non-rectilinear drawings of K_n [10,13]. As the catalogue grows, exact values for $\overline{\mathrm{cr}}(K_n)$ will be found. The catalogue is being extended computationally by [6] Dean. Additionally, Thorpe and Harris [22] have accomplished a randomized search and produced drawings of K_{12} and K_{13} with 155 and 229 edge crossings, respectively. Both drawings have fewer edge crossings than the drawings given by Jensen [15]. Our question is the following: how many non-equivalent drawings of K_n produce a number of edge crossings equal to $\overline{\mathrm{cr}}(K_n)$? Experimental work leads us to believe that the answer to this question is nontrivial. As more concrete information becomes available, we will be better able to investigate this question. Lastly, we note that Brodsky et al. [5] have given a combinatorial proof that $\overline{\mathrm{cr}}(K_{10})=62$. We know of only one drawing of K_{10} with 62 edge crossings.

Our third and final open question concerns a problem addressed by Hayward [14] and Newborn and Moser [17] and is the following: find a rectilinear drawing of K_n that produces the largest possible number of crossing-free Hamiltonian cycles. Hayward, building on the work in [17], has asymptotics based on a generalized rectilinear drawing of K_n, as mentioned in Section 4, Table 1. Our construction given in Section 5 improves Hayward's result. A related open problem is: does some rectilinear drawing of K_n with the minimum number of edge crossings necessarily produce the optimal number of crossing-free Hamiltonian cycles? Hayward conjectures that the answer is "yes", as do we, but as of yet, no proof is known.

Crossing number problems are rich and numerous with much work to be done. For an excellent exposition of further diverse open questions, see [19].

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