RESTRICTED 2-FACTOR PROBLEMS ARISING FROM MOMENT-BASED POLYGON RECONSTRUCTION *

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1. Problem Definition. The problem of reconstructing an unknown polygonal shape P, viewed as a region in the complex plane, from a finite number of its complex moments is motivated by a number of mathematical, computational and application-oriented considerations. Complex moment information can be derived from such diverse physical processes as tomographic (line integral) measurements and measurements of the exterior gravitational or magnetic field or the thermal radiation associated with an otherwise unspecified object [5, 8, 9]. Milanfar *et al.* [8], building on earlier work of Davis [1, 2], show how a finite number of complex moments of P can be effectively used to reconstruct the vertices of P (and hence P itself if sufficient additional information–for example, convexity–about P is known). Very recently, Golub *et al.* [5] address some of the formidable numerical difficulties associated with this reconstruction. In addition, they demonstrate that partial information concerning a polygon's edges can also be derived from a finite number of its complex moments, raising the possibility of a more complete and general reconstruction scheme. (In general, even simply-connected polygonal regions are not uniquely specified by even infinitely many of their complex moments [9], thereby precluding a completely general reconstruction scheme.)

Given vertex positions for the set V of vertices of P in the Cartesian plane and partial information (expressed in terms of geometric constraints) about the edges bounding P, we examine the problem of constructing polygonal regions consistent with this information. Although in practice, the vertex positions and edge constraints may contain error in their specifications (due to error either originating in the data or introduced in the numerical computations), it is of interest to study the reconstruction problem under the assumption that the moment data and its processing are error-free. As we shall see, this translates to a (geometrically) constrained directed 2-factor problem (cf. [7]) in a digraph G defined on the vertex set V.

As observed in [5], the potential edges incident on a vertex are subject to very specific local restrictions. These restrictions have the following simple geometric interpretation. Each vertex has two independent axes assigned to it: a blue and a red axis. Each axis specifies two in-directions and two out-directions such that in-directions and out-directions are orthogonal (see Figure 1).

An edge joining vertex v to vertex w belongs to the edge set of G if and only if its direction agrees with one of the axes specified at each of v and w. We will colour the head and tail of each such edge with the colour of its associated axis. A solution to the polygon reconstruction

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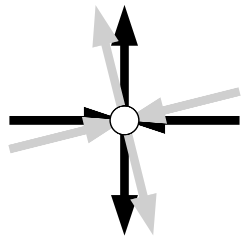

FIG. 1. *local blue and red axes for potential edges at a vertex*

problem is a spanning subgraph H of the resulting directed and edge-bicoloured graph in which every vertex has both in-degree and out-degree one and both red-degree and blue-degree one. Thus we are looking for a directed 2-factor of G that satisfies some local colour constraints. Of course, known properties of the original polygonal region (for example, that it is convex, rectilinear or simply connected) impose additional constraints that affect the complexity of the reconstruction task. In particular, when the input polygon is rectilinear (all edges are horizontal or vertical) it is easily demonstrated that the two colour axes coincide, which means that the colour constraints disappear entirely!

2. Colour-Constrained 2-Factor Problem. It is well known that the unconstrained 2-factor problem is polynomial-time solvable [4, 7]. Our problem differs from the unconstrained 2-factor problem simply by the addition of colour constraints on the choice of edges. If we ignore the geometric source of the colour constraints, our problem reduces to the following:

BICOLOURED DIRECTED 2-FACTOR
INSTANCE: Directed graph $G = (V, E)$, where edges of E are coloured (red or blue) at each end.
QUESTION: Does G admit a directed 2-factor H with red-degree and blue-degree one at each vertex?

Using a component-based proof, we give a reduction from **3-DIMENSIONAL MATCHING** [4] to **BICOLOURED DIRECTED 2-FACTOR**. The problem **BICOLOURED DIRECTED 2-FACTOR** remains **NP**-hard even if every vertex in V has red in-degree, red out-degree, blue in-degree, and blue out-degree at most two (reflecting the degree constraints inherent in our reconstruction problem). In fact, the problem remains **NP**-hard even if just one of the coloured directed degrees at each vertex is at most two and the others are at most one. Further restriction to all coloured directed degrees at most one at every vertex produces a problem reducible to **2-SAT** (and hence in **P**).

3. Non-Crossing 2-Factor Problem. Since the edges bounding a simple polygonal region do not intersect except at their endpoints, it is natural to impose a non-crossing constraint on the edges of our reconstructed polygon. If we ignore the colour constraints this yields the following natural problem:

NON-CROSSING 2-FACTOR
INSTANCE: Directed embedded graph $G = (V, E)$.
QUESTION: Does G admit a 2-factor H, none of whose edges intersect except at their endpoints?

We give a component-based reduction from **3-DIMENSIONAL MATCHING** to **NON-CROSSING 2-FACTOR**. The problem **NON-CROSSING 2-FACTOR** remains **NP**-hard even if the graph G has bounded degree.

4. Existence of Exponentially Ambiguous Solutions. While the fact that polygonal regions are not uniquely reconstructible from their convex moments implies the existence of distinct polygonal regions with identical moments, the actual construction of such ambiguous regions, especially simply connected examples, is by no means straightforward. An example due to Strakhov and Brodsky [9] with 22 vertices remains (to our knowledge) the smallest ambiguous simply connected example known. Despite this we have been able to construct a family of examples, based on the Strakhov/Brodsky construction, which demonstrate exponential ambiguity: the number of distinct simply connected regions consistent with a fixed set of moments grows exponentially with the size of the regions.

5. Conclusions. We have demonstrated that two natural generalizations of the general 2-factor problem are **NP**-hard. In practice, however, a reconstruction algorithm based on contraint propagation performs remarkably effectively even in the presence of numerical error arising in the processing of exact moment information [3]. Enumeration of ambiguous solutions is achieved through recursive branching along ambiguous search paths. A challenge arises in the reconstruction of polygonal regions (or fragments of the boundary of such regions) whose vertex and edge information is only partly valid. Here we require heuristic assumptions in identifying various degrees of reliability within the reconstructed information. In addition, we anticipate being able to improve the numerical estimates generated in the first phase of the reconstruction process by exploiting some of the combinatorial structure uncovered in the second phase.

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