Optical Flow

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Constraint equation…

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- One equation per pixel, two unknowns…

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- One equation per pixel, two unknowns…
- Look at neighborhood!

Constraint over n pixels

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$$-\frac{\partial E}{\partial t} = \frac{\partial E}{\partial x}\frac{dx}{dt} + \frac{\partial E}{\partial y}\frac{dy}{dt} \Rightarrow -E_t = E_x u + E_y v$$

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region to be used

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- Given n pixels, then we have

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- How to solve??

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region to be used, generally 2x2

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- How to solve?? Least squares!

Least squares..

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$$Ax = b$$

$$x = \left(A^t A\right)^{-1} A^t b$$

Least squares..

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- Inversion at each pixel is SLOW

Eigen values

$$x = \left(A^t A\right)^{-1} A^t b$$

- Eigen values of A^tA here give insight into the nature of the least squares results.
- Two eigen values, $\lambda_{1,}$ $\lambda_{2,}$ where 1 is largest.

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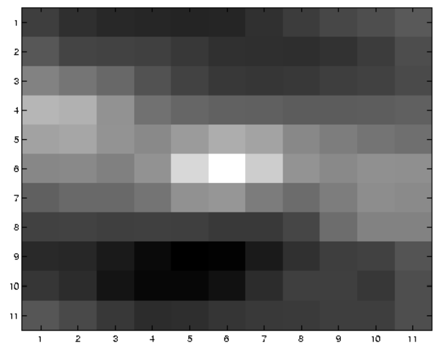
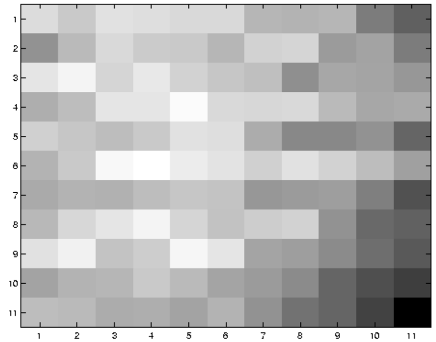
Eigen values

- If… $\lambda_1, \lambda_2 > \tau$ (threshold) – unique solution, do inverse.



Eigen values

- If… $\lambda_{1,}$ λ_2<= τ – no strong gradient info (flat)



Eigen values

- If… $\lambda_1 > \tau$, $\lambda_2 <= \tau$ – constant gradient. Use gradient flow

http://graphics.cs.cmu.edu/courses/15-463/2005_fall/www/Lectures/direct-alignment.ppt

Lets solve it!

$$-E_t = E_x u + E_y v$$

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$$\left\langle E_{x1}, E_{y1}, E_{t1} \right\rangle$$

$$\left\langle E_{x2}, E_{y2}, E_{t2} \right\rangle$$

$$\left\langle \cdots, \cdots \right\rangle$$

$$\left\langle E_{xn}, E_{yn,} E_{t1} \right\rangle$$

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Algorithm!

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4. Invert / ignore /gradient flow

Get eigen values

- What is an eigen value?

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$$(A - \lambda I)x = 0$$

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In our case, 2x2 window…

$$A - \lambda I = \begin{bmatrix} A_{11} - \lambda & A_{12} \\ A_{21} & A_{22} - \lambda \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Get eigen values

$$(A - \lambda I)x = 0$$

This MUST be invertible, to solve for x.

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$$\det(A) = ad - bc = (A_{11} - \lambda)(A_{22} - \lambda) - A_{12}A_{21}$$

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Algorithm!

1. Compute Ex, Ey, Et
2. Compute Ex^2, Ey^2, ExEy, ExEt, EyEt
3. **Convolve step 2 with gaussian weights**
4. Check if AtA is invertible (look at eigen values)
4. Invert / ignore /gradient flow